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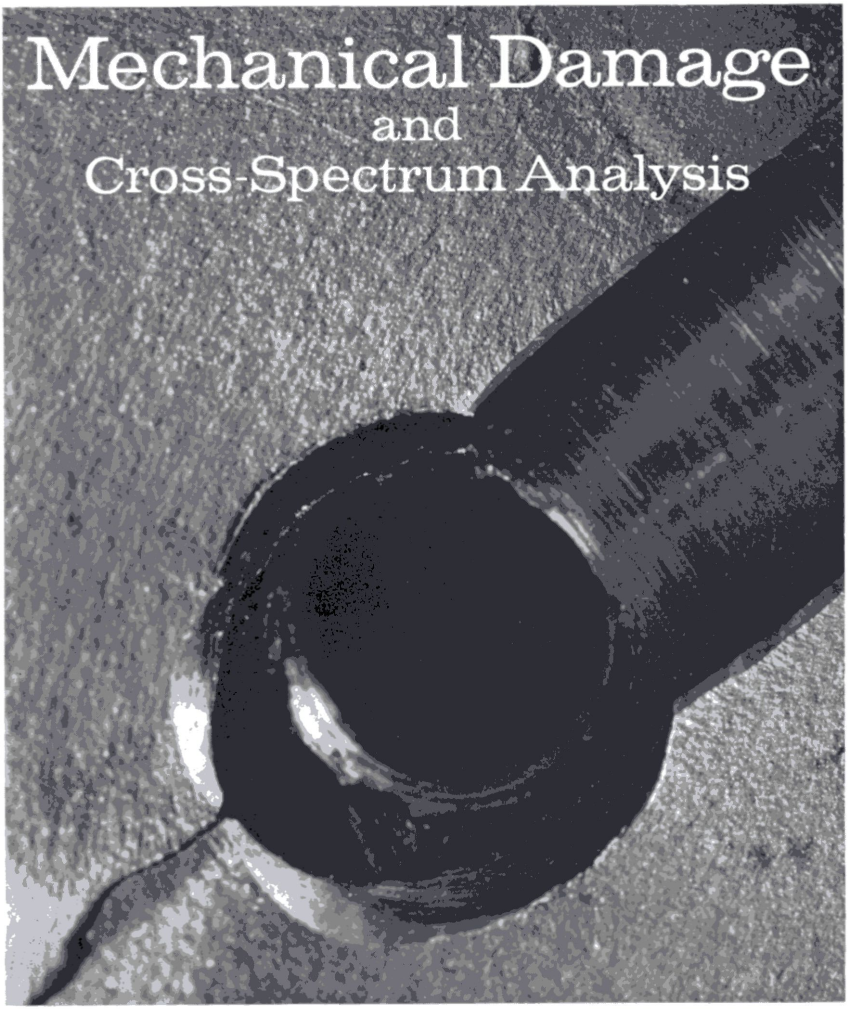

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On the Damaging Effects of Vibration

by

Jens T. Broch, Dipl. ing. E.T.H.*)

ABSTRACT

It is a well known fact that materials subjected to vibrational stresses eventually fatigue, which, in certain cases, may cause catastrophic failure to occure. The physical teory underlying fatigue phenomena is still not completely understood even though the concept of fatigue failures has been known (and investigated) for more than 100 years.

In this article a brief survey is given of some of the facts which constitute the basis for theoretical fatigue life estimates today. Both periodic and non-periodic (random) stress-versus-time relationships are considered.

It is shown that to obtain a reasonably accurate estimate it is in general necessary to combine theoretical predictions with practical vibration test experiments. The importance of dangerous dynamic stress concentrations, nonlinear resonances and interaction between resonances are discussed, and methods of taking these effects into consideration in practical fatigue life predictions are suggested.

Finally, mechanical failure caused by other vibrational effects than fatigue are briefly touched upon.

SOMMAIRE

C'est un fait bien connu que les matériaux soumis à des efforts de vibration finissent par fatiguer, ce qui dans certains cas peut provoquer des défectuosités catastrophiques. La théorie physique fondamentale des phénomènes de fatigue n'est toujours pas saisie complètement, bien que la nation de défectuosités résultant de la fatigue soit connue (et étudiée) depuis plus de 100 ans.

Dans cet article est donné un bref aperçu de certaines des réalités qui constituent aujourd'hui la base des esimations de la durée de vie théorique à la fatigue. On considère les relations par rapport au temps d'efforts tant périodiques que non-périodiques (aléatoires).

On montre que pour obtenir une estimation raisonnablement précise, il est généralement nécessaire de combiner les prédictions théoriques avec des expériences d'essais pratiques aux vibrations. On discute de l'importance de concentrations de contraintes dynamiques dangereuses, de résonances non-linéaires et d'interactions entre résonances et l'on suggère des méthodes de prise en considération de ces effets dans les prédictions de durée effective de vie à la fatique.

Finalement sont brièvement traitées les défectuosités mécaniques consées par d'autres effest vibratoires que la fatigue.

ZUSAMMENFASSUNG

Bekanntlich ermüden schwingungsbeanspruchte Materialien im Laufe der Zeit, was in bestimmten Fällen Ursache für katastrophales Versagen sein kann. Die dieser Ermüdungserscheinung unterlegte physikalischen Teorie ist noch nicht restlos geklärt, obwohl der Begriff des Versagens infolge Materialermüdung seit mehr als 100 Jahren bekannt ist (und Gegenstand von Untersuchungen ist).

In diesem Artikel wird eine kurze Übersicht auf einige Punkte gegeben die heutzutage die Grundlage zur theoretischen Abschätzung der Dauerfestigkeit bilden. Periodische und nichtperiodische (stochastische) Schwingbeanspruchungen werden in ihren Zeitbeziehungen betrachtet.

Es wird gezeigt, daß die theoretischen Vorhersagen normalerweise mit praktischen Schwingprüfversuchen kombiniert werden müssen, um einigermassen genau abzuschätzen. Die Wichtigkeit der gefährlichen dynamischen Spannungskonzentration, nichtlineare Resonanzen

*) Tutorial lecture presented in Kungälv, Sweden, on the 23rd of October 1968.

und die Wechselbeziehung zwischen den Resonanzen werden diskutiert, und Methoden werden vorgeschlagen, wie diese Einflüsse bei der praktischen Vorhersage der Dauerfestigkeit in Betracht gezogen werden können.
Schließlich werden in knappen Worten andere Schwingungseffekte angedeutet, die außer Materialermüdung zum mechanischen Versagen führen.

Introduction

When considering the effects of vibrations on materials and mechanical components one of the first thoughts which comes to mind is that of mechanical fatigue. This is quite natural as mechanical fatiguing effects are definitely important in estimating the "life" of a particular construction subjected to vibrations.
However, mechanical fatigue is not the *only* deteriorating effect of vibrations, and the construction may fail in practice for other reasons as well. Failure may, for instance, be caused by the occurrance of one, or several, excessive vibration amplitudes (contact – failures, collisions), or by the fact that a certain vibration amplitude value is exceeded in too great a fraction of time (focusing in optical systems).
If one no longer considers the mechanical construction as a whole, but rather the strength of the various materials used in the construction, mechanical fatigue will certainly be of prime importance. Since the construction of the various mechanical parts rests upon a certain knowledge as to how the materials used behave under the kind of loading expected to occur in practice, it seems reasonable to begin this lecture with a few words on the concept of mechanical fatigue in engineering materials.

The Concept of Mechanical Fatigue

Even though the term "fatiguing of engineering materials" was used already in 1839 by the French professor I. V. Poncelet, the concept of mechanical fatigue seems first to be taken into more universal use in literature around 1855.
Some of the first thoughts on the subject which is now commonly termed fatigue fracture were at that time introduced in connection with fractures in railroad car axels. The ideas were rather primitive and built upon an incorrect understanding of the atomic structure of the material. A systematic quantitative investigation of the relationship between vibrational quantities and the fatigue life of the material being vibrated does not seem to have appeared until the very impressive work carried out by Wøhler was presented in 1858. Here, for the first time, systematic scientific research was used to attack the very important engineering problem of fatigue failures.
One of Wøhler's most important, practical conclusions was that if the maximum amplitude of the vibrations stayed below a certain limit value the vibrations would not cause fatigue failure at all. It is now known this is only true for certain engineering materials, and even then only to a certain extent. Nevertheless Wøhler's above mentioned design "criterion" has, in the past many

years, been widely used by mechanical engineers all over the world, and the importance of his work cannot be overemphesized.

The graphical presentation of the kind of data obtained by Wøhler, the so-called "Wøhler-curve", still forms the basis for theoretical estimations of the time to fatigue failure of a material. The same curve, or rather the same type of curve, is usually, in Anglo-American literature, called the S-N curve (*S*tress vs. *N*umber of cycles to failure). An example of such a curve valid for 4340 steel is shown in Fig. 1.

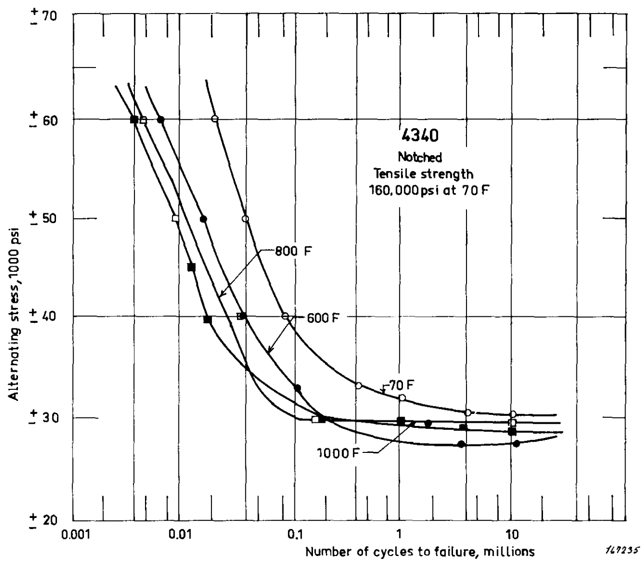

Fig. 1. Fatigue strength curves for notched 4340 steel (From Metals Handbook).

The ordinate in the figure represents the maximum vibration stress amplitude (peak value) while the number of stress reversals to failure is indicated on the abscissa. The curves shown have been obtained from tests with zero mean stress, i.e. they refer to purely sinusoidal loading of the material.

Without going into details some recent findings on the physical nature of the fatigue of metals are given in the following. The findings cited have developed mainly from the field of aircraft material research where fatigue problems are a major concern.

The *fatigue phenomenon* is today deemed to originate from local yield in the material or, in other words, from *a sliding of atomic layers.* This sliding is caused by a combination of so-called "dislocations" (irregularities in the crystalline structure of the material) and local stress concentrations. It is now assumed that each slip, no matter how small, is connected with a small deterioration of the material, independent of the direction of the slip. The deterioration stops only when the slip stops. Some definite proof for this hypothesis has, to the author's knowledge, not been established as yet (Sept. 1968). It gives, however, a logical and reasonable explanation for the formation of the microscopic "slip bands" which is the first visible sign of material fatigue.

When slip bands have been formed they are, under continuous vibration loading, observed to progress and form minute cracks which eventually join together and produce major cracks. As soon as a crack has reached a certain size it will propagate through the material according to a mathematical law of the form:

$$\frac{dx}{dN} = c\, e_r^{m}\, x^{n} \tag{1}$$

where x = crack length
N = number of stress reversals
c = constant, dependent upon the material properties (a reasonable assumption seems in many cases to be $m = 2$, $n = 1$).
e_r = relative strain.

Finally the crack will become so large that the stress in the remaining material becomes too great, whereby the crack propagation becomes unstable, and fatigue failure occurs.

Even though it is possible to describe a certain part of the fatiguing process by means of a relatively simple mathematical expression, (1), both the forma-

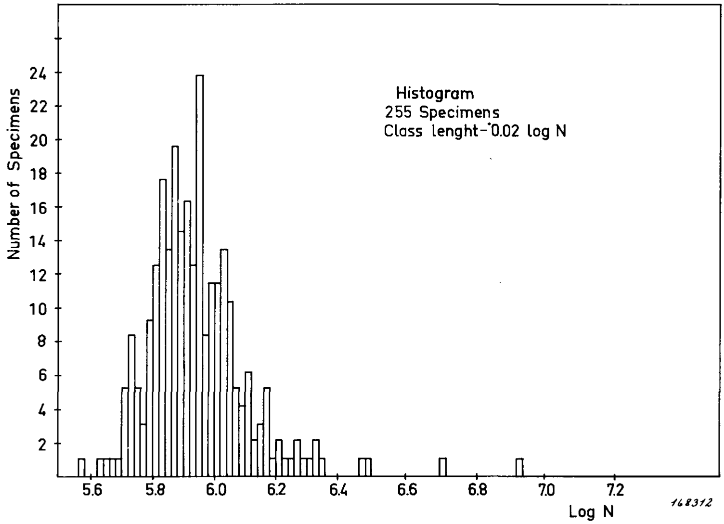

Fig. 2. Typical histogram obtained from fatigue experiments (after Bloomer and Roylance).

tion of "slip bands" and the final crack instability stages are of a highly statistical nature. Taken as a whole, therefore, fatigue failures must be regarded as statistical phenomena as suggested by Weibull in the nineteen forties.
The statistical nature of the phenomenon reveals itself in a considerable spread in the results of fatigue experiments. As an example of the result of such experiments Fig. 2 shows a histogram made by Bloomer and Roylance from investigations on the fatigue life of notched aluminium specimens. The results shown were obtained from tests at a single vibration stress level, and thus represent a single point on the corresponding S-N-curve. From this it is easily visualized that the determination of the S-N-curve for a particular engineering material is an extremely time – consuming procedure if a reasonably high statistical significance is required from the data. It may, in this connection, be mentioned that to employ normal "simple" statistical methods to determine the significance of the measured results, use should be made of roughly 50 specimens per test level (i.e. per data point)!
The number of specimens used to obtain Fig. 2 was 255. The reason for the large number specimen was, in this case, not to determine a point on the S-N-curve with an extremely high statistical significance, but to try and carry out a more detailed statistical analysis of the distribution function. It is, for instance, possible to determine the mathematical law underlying the process from the shape of the distribution function. This can be done by trying different mathematical transformations of the histogram's X-axis (Fig. 2) until the histogram fits a normal (Gaussian) probability density curve.
Another method consists in using probability paper when plotting the measured data, Fig. 3. From Figs. 2 and 3 it appears that the fatigue life of the specimens in this case seems to follow a logarithmic law at the vibration level used for the tests.
On the basis of statistical analysis of the above mentioned type, together with microscopic investigations of the fractures an hypothesis thas been formed that two different fatigue "mechanisms" govern the fatigue life of a particular material. One of these "mechanisms" is supposed to be active at low stress levels while the other acts at high stress levels. The transition from one "mechanism" to the other takes place gradually with stress level. A fact which seems to support this theory is that most engineering materials show a more or less distinct "bend" in the S-N-curve at a certain stress level, see Fig. 2.

Estimation of the Fatigue Life of a Material

When the S-N-curve for a given material is known the number of stress reversals of pure harmonic vibration with a certain maximum amplitude (peak value) required to produce fatigue failure can be read directly off the curve. Or rather, it is possible to read off the curve *the average number of stress reversals to failure* as the spread in fatigue life of the individual specimens is quite considerable, cfr. the above discussion and Figs. 2 and 3. It is possible, however, when the statistical distribution of the fatigue lives is known, to

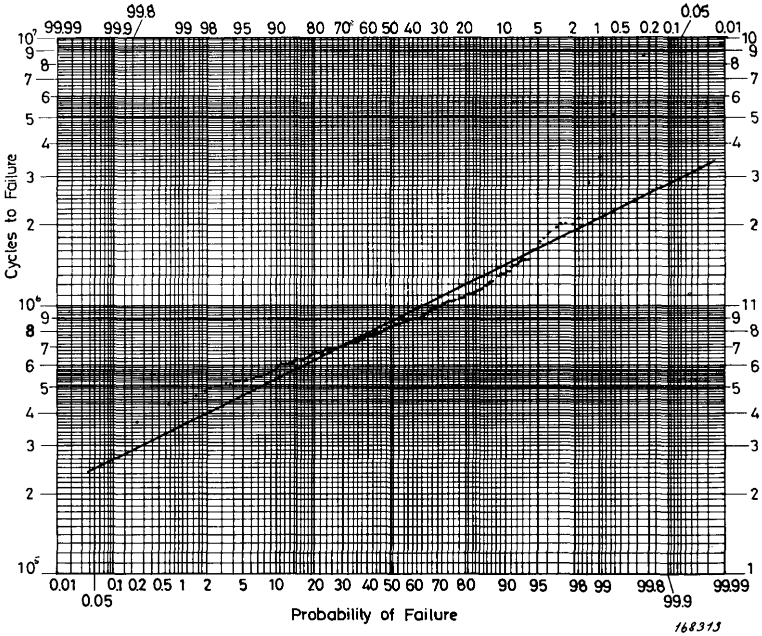

Fig. 3. Example of the use of probability paper: Log-normal probability plot of the data Fig. 2.

estimate the probability of failures for any given number of stress reversals. On the other hand, in practice a mechanical part, or material, is only very rarely, if ever, subjected to pure harmonic vibrations of constant maximum amplitude. The question thus remains: Is it possible also in practical cases of heavy dynamic loading, to estimate the fatigue life of a material. The answer to this question must be a somewhat reluctant: "Yes, to a certain extent". From a scientific point of view the problems involved have not yet been solved, and a number of investigations, theoretical as well as experimental, remain to be done before a definite quantitative description of the mechanical fatigue phenomenon can be formulated.

On the other hand, several empirical "rules" for fatigue life predictions, suitable for engineering use, have been proposed by a number of research workers. The first time that such a "rule" seems to have been suggested is in 1924 where A. Palmgren published a paper on "Die Lebensdauer von Kugellagern". The same "rule" was, independently, reestablished in 1937 by B. F. Langer and in 1945 by M. A. Miner. In anglo-american literature the "rule" is commonly termed "the Palmgren-Miner Rule" or simply "Miner's Rule". It states that fatigue fracture is the result of a linear accumulation of "partial" fatigue damages. If for example, a material is subjected to a certain number of stress reversals, n_i, at a certain vibration level, and the total number of stress reversals to failure at this level is N_i, then the material has attained a "partial" fatigue damage of

$$D_i = \frac{n_i}{N_i} \tag{2}$$

If the vibration level is changed a new "partial" fatigue damage figure can be calculated for this new level (N_i is found from the material S-N-curve), and the totally accumulated fatigue damage after vibrations at different levels is:

$$D = \Sigma \frac{n_i}{N_i} \tag{3}$$

Failure occurs when $D = 1$.
It is not difficult to find weaknesses in the above hypothesis. On the other hand, it presents a mathematically simple method to estimate the fatigue life of a material. Although various suggestions have been made over the past twenty years in order to improve the "rule", these improvements have always turned out to have limited validity only. It seems reasonable, therefore, to base engineering fatigue life estimates on (3) until a scientifically saticfactory, quantitive description of the fatigue phenomenon has been established. That the result of such fatigue life calculations must be regarded as very rough estimates only is, in view of the above discussion, obvious.

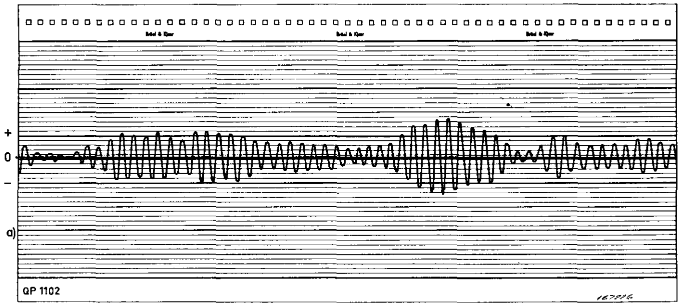

Fig. 4. Typical strain-versus-time trace obtained from measurements on a structural member subjected to narrow band random vibrations.

By using the expression (3), and a mathematical approximation to the S-N-curve it is sometimes possible to establish a closed mathematical formula for D. This will be exemplified later in the text. Two conditions which have to be fulfilled when use is to be made of (3) and the ordinary S-N-curve are, however, that each stress reversal has an approximately sinusoidal wave-shape and

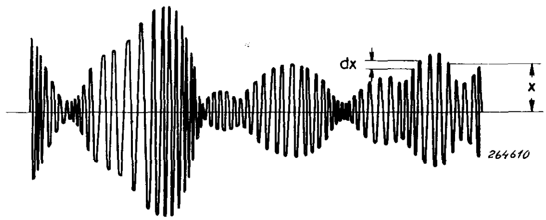

Fig. 5. Illustration of the concepts leading to the expression, n(x), for the number of strain (stress) peak values which fall within the amplitude level, x.

that the mean stress is zero. These conditions are fulfilled, for instance, by the vibrations shown in Fig. 4. Stresses of the type shown here occur in single degree-of-freedom mechanical systems excited by Gaussian random vibrations. As this is a relatively important practical case of dynamic loading it might be worth while to try and establish the corresponding mathematical expression for D.

Assuming now that the vibration can be considered to be stationary it is possible to describe the occurance of maximum amplitudes (peak values) in the form of probability data. From the figure (Fig. 4) it can furthermore be seen that the average vibration frequency, i.e. the average number of stress reversals per second, may be considered constant, say f_o. The total number of stress reversals in the time T then becomes $n_T = f_o T$, and the number of stress reversals, the maximum amplitudes of which fall within a small amplitude interval, dx, around the amplitude value, x, is (see Fig. 5):

$$n(x) = f_o T \, p(x) \, dx$$

where $p(x)$ is the peak probability density function.

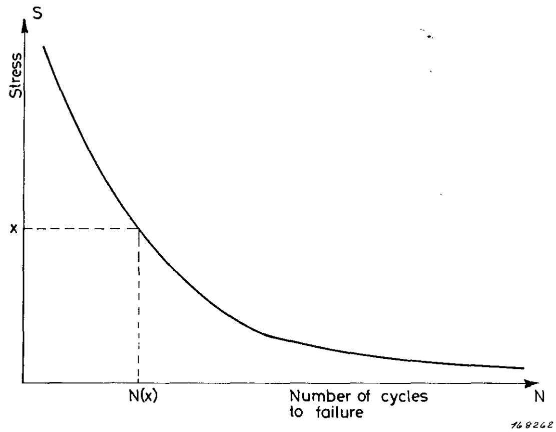

Fig. 6. Illustration of the connection between the stress amplitude, x, and the average number of stress reversals, N(x), to failure at this stress level.

The partial fatigue damage caused by these stress reversals is, according to the Palmgren-Miner Rule (2):

$$D_x = \frac{n(x)}{N(x)} = f_o T \frac{p(x)dx}{N(x)} \tag{4}$$

$N(x)$ is, as pointed out above, found from the S-N-curve for the material, see Fig. 6.
By summing the partial fatigue damages for all values of x one finds:

$$D = \sum_{o}^{\infty} D_x = \sum_{o}^{\infty} \frac{n(x)}{N(x)} = f_o T \int_{o}^{\infty} \frac{p(x)}{N(x)} dx \tag{5}$$

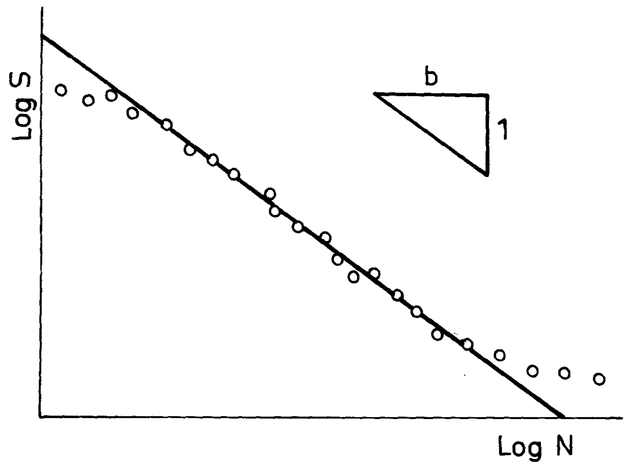

Fig. 7. Approximation to the S-N-curve by the expression $NS^b = a$.

Fatigue failure occurs when $D = 1$ whereby the total average time to failure becomes

$$T = \frac{1}{f_o \int_{o}^{\infty} \frac{p(x)}{N(x)} dx} \tag{6}$$

The formula (5) and (6) may be considered generally valid for stress-variations of the type shown in Fig. 4 (if one accepts the validity of the Palmgren-Miner Rule!). *To allow a fatigue life estimate to be made under these conditions it is thus only necessary to know the S-N-curve for the material, the peak probability density curve for the stress variations, and the average vibration frequency.*

If both the functions $N(x)$ and $p(x)$ can be expressed analytically the average fatigue life of the material can be estimated by performing the integration (6) either directly or by means of numerical methods. *In cases where $N(x)$ and $p(x)$ are available in the form of experimental curves only, these may either be approximated analytically and inserted in (6), or the expression $p(x)/N(x)$ must be calculated, the result plotted as a curve, and the curve be graphically integrated.*

In this connection it should be pointed out that for many engineering materials it is possible to approximate the "dangerous part" of the S-N-curve by means of an expression of the type

$$\underline{NS^b = a} \qquad (7)$$

where N is the Number of stress reversals to failure at the stress level, S, see Fig. 7. a and b are constants which depend upon the material. In the case of common engineering materials b takes values between 3 and 8 (Steel: $b \approx 3.5$, Tinbronze [80 % Cu – 10 % Sn – 10 % Pb]. $b \approx 7.5$). The formula (7) is in good agreement with the logarithmic fatigue life distribution discussed in connection with Figs. 2 and 3.

Fatigue Damage in Mechanical Constructions

A mechanical construction consists, in general, of a number of mechanical "elements", such as plates, beams, bolts etc. Each of these elements may be produced from a different material and show a variety of specialized shapes. To theoretically estimate the fatigue life of the complete construction is therefore extremely difficult, if at all possible. In some cases the only solution may be to perform fatigue life testing of the complete construction in the laboratory by means of modern vibration test equipment. Considering, however, the spread in fatigue life discussed earlier in the text, even this solution is not very suitable in practice because a large number of complete constructions need to be tested. It is therefore normally necessary to establish some sort of engineering estimate upon approximate calculations and a "reasonable" number of complete vibration tests.

Such approximate calculations are often based upon the designer's own experience, and his ability to discover the "weak points" in a construction. The latter is very important as *it is the weakest link in a chain which determines the total strength of the chain.*

Regarding "weak points" in a mechanical construction subjected to vibrations, the primary objective will be to locate points with high dynamic stress concentration and to be aware of dangerous resonances.

An experienced designer is very often able to rapidly locate points with high stress concentration in a construction subjected to a known static loading. When the construction is subjected to dynamic loads, however, the location of points with high stress concentrations may be an entirely different matter. Here resonance phemomena may cause dangerous stress concentrations to occur at points which, during static loading, show only very low or moderate stresses.

Also, resonant nonlinearities may cause "unexpected" excitation of other resonances.

If it is not possible, from purely theoretical considerations, to determine the response at various points when the construction is subjected to vibrations, it is necessary to carry out vibration tests in the laboratory. In carrying out these tests use should be made of socalled "frequency sweep tests", using pure sinusoidal vibrations. It is then possible to study the response at the different points directly as a function of frequency. The tests are quite time consuming as normally the vibration response of a large number of points must be investigated for several methods of mounting of the test specimen (the complete construction). They also require considerable technical know-how from the engineer in charge of the testing, although some help may be gained by the use of so-called "stress coating" and stroboscopic observations. However, upon conclusion of the tests one will, with a fair degree of certainty, have located the points on the construction where fatigue damage is most likely to occur.

The next "on the agenda" is to consider to what kind of vibrations the construction would be subjected during normal operation: Periodic vibrations, "nearly periodic" vibrations, random vibrations, shock excitations, etc. If it is not possible to theoretically calculate the response at the "critical points" to the expected kind of vibration excitation, one must, again, carry out vibration experiments. The experiments will now not be nearly as time consuming as before, in that now *both* the "critical points" on the construction *and* the most dangerous loading condition are known. The greatest problem will therefore often be to simulate the practical vibration excitation with sufficient accuracy. When this problem is solved it remains only to *determine the peak probability density curve, and the average frequency of the dynamic stresses at the "critical points".*

From a knowledge of the "critical points" on the construction it is obvious that also the materials used at these points are known. If the S-N-curve for the materials are available (from material handbooks) it is a relatively simple matter to estimate fatigue damage according to formula (5) and thereby also to estimate the expected average fatigue life of the construction, *when the stresses are of the type illustrated in Fig. 4.*

The latter condition, i.e. that the stresses are of the type shown in Fig. 4, is, unfortunately, not always fulfilled in practice. Very often the dynamic stresses at the "critical points" show a stress-versus-time dependency such

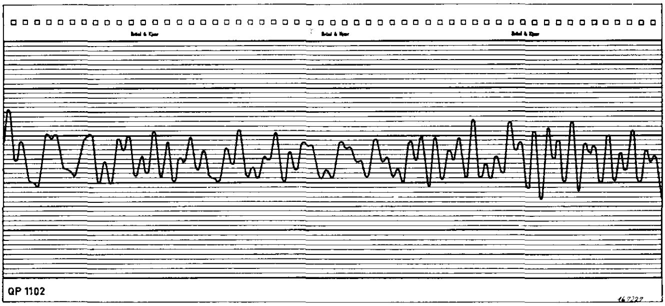

Fig. 8. Typical stress-versus-time trace at a "critical point" on a complicated structure.

as recorded in Fig. 8, corresponding to a frequency response spectrum of the type shown in Fig. 9. The reason is, as can be seen from Fig. 9, that, *in general, several resonances in the construction interact.* When the construction is subjected to wide-band random vibrations this results in stress-versus-time relationships of the type indicated in Fig. 8.

Now, how may the average fatigue life of the construction be estimated when the dynamic stresses at the "critical points" are of the type illustrated in Figs. 8 and 9? This question is, at least at present, not so easily answered. Investigations are, for the time being, in progress in the U.S.A., as well as in Europe, to try and clarify the problems involved. Some experiments in this direction were made at Brüel & Kjær in 1967 and the results are published in the Brüel & Kjær Technical Review No. 1-1968. Based on these results a brief outline of the state-of-the-art is given in the following.

The general idea behind experiments of the kind described in the above mentioned Brüel & Kjær Technical Review (No. 1-1968) is that a comparison

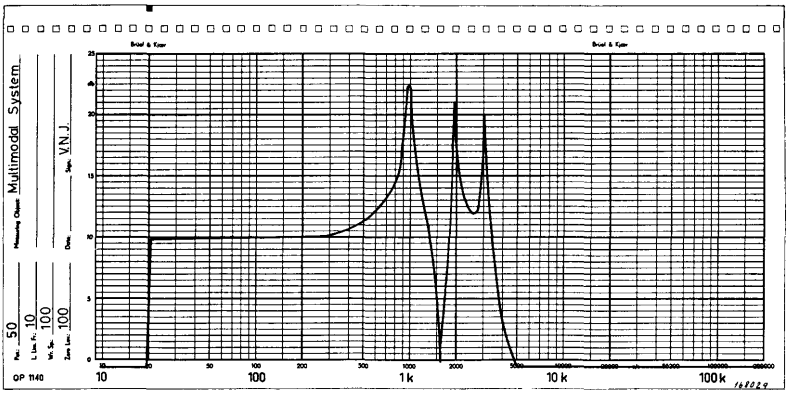

Fig. 9. Frequency spectrum of the stress illustrated in Fig. 8.

is made between the fatigue life of a linear single degree-of-freedom system excited by Gaussian random vibrations, and the fatigue life of a slightly more complicated system under the same loading conditions (random vibrations). An essential part in the setting up of such experiments is, however, that *only one system parameter should be changed from experiment to experiment.*
The reason for choosing a linear single-degree-of-freedom system as "reference" system is that the average fatigue life of such a system under random loading may be estimated theoretically from formula (6), assuming that the Palmgren-Miner hypothesis is valid. The peak probability density curve for the stresses is in this case given by the expression

$$p(x) = \frac{x}{\sigma^2} e^{-\frac{x^2}{2\sigma^2}} \quad (8)$$

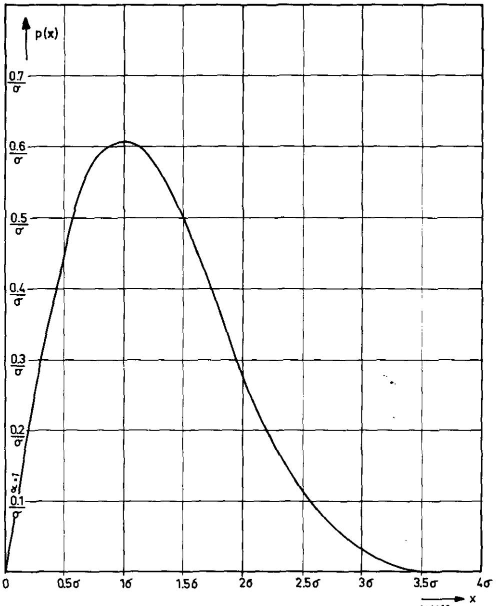

Fig. 10. Typical peak probability density curve for narrow band random vibrations (Rayleigh distribution).

where σ is the RMS (root mean square value) of the stress. Fig. 10 shows a plot of $p(x)$ with σ as parameter and the resulting curve is named the Rayleigh probability density curve. Utilizing the expression (7) as an approximation to the S-N-curve one obtains by means of (6), (7) and (8):

$$T = \frac{a}{f_o (\sqrt{2}\,\sigma)^b\, \Gamma\left(1 + \frac{b}{2}\right)} \qquad (9)$$

where Γ is the gamma-function. Thus when f_o, a, b and σ are known it is an easy matter to calculate T.

Returning now to the experiments referred to in the Brüel & Kjær Technical Review No. 1-1968, the average fatigue life of a single degree-of-freedom system was here compared to the average fatigue life of a two degree-of-freedom system. Both systems were made from the same material and were subjected to gaussian random vibrations producing the same RMS-value (σ) of the stress, and the same average frequency, at the critical point. It was found that *the average fatigue life of the two degrees-of-freedom system was considerably longer than the average fatigue life of the single degree-of-freedom system.*

Unfortunately it is not possible on the basis of known experimental results, to establish a firm quantitative design formula for fatigue life estimations in complicated cases of dynamic loading. One conclusion which may, however, be drawn from reported experimental results, and their theoretical interpretations, is that *when the dynamic stresses at the critical point in a construction is of the Gaussian random type, fatigue life estimates according to formula (9) above will indicate the shortest average fatigue life that is to be expected.*

To perform such a "rough" calculation the only vibrational quantities needed are the RMS-value of the stress and the average vibration frequency (number of zero-crossings) at the critical point.

A measurement of the average vibration frequency can be made simply by means of an electronic counter. If, on the other hand, only the frequency spectrum and the RMS-value of the dynamic stresses are known then the average vibration frequency can be calculated according to a method also described in the above mentioned Brüel & Kjær Technical Review No. 1-1968.

Many research workers in the field of mechanical fatigue today question the validity of the Palmgren-Miner hypothesis. Because very often in practice the dynamic stresses are of a stochastic (random) nature it has been suggested that *S-N-curves should be obtained from tests using narrow band vibration excitation* (Fig. 4) instead of the present practice of using sinusoidal excitation. Corrections for stress-versus-time traces of the kind shown in Fig. 8 could then be made on the basis of actual peak probability density (and frequency spectrum) information.

Functional Failures Caused by Vibrations

As the concept of functional failure has a very broad coverage, and since the possibility of a functional failure depends greatly upon the particular construction, it is only possible in the following to outline some more general views on the matter, such as:

a) Functional failure caused by the occurance of a single (or several) very large vibration amplitudes, and
b) Functional failures caused when the displacement amplitude exceeds a certain, critical value for too great a fraction of the time.

As mentioned in the beginning of this lecture functional failures of the type a) above may be caused by contact-errors or chattering phenomena in relay contacts, or by collision between two elastically mounted masses. A third example of functional failures, caused by too large displacement amplitudes, is electrical short circuiting, a failure which, by the way, is not uncommon. An exact calculation of the time elapsing between the moment the construction is subjected to vibration and the *first* time the displacement amplitude reaches a critical value, requires a thorough knowledge of the vibration-versus-time relationships. In by far the most practical situations such a calculation will not be possible. One may, in special cases, introduce simplifying assumptions and try to make a rough estimate of the "danger". On the other hand, such estimates are normally not very reliable and may thus be of very limited value in practice.

When the vibrations considered are of a Gaussian random nature it is possible to calculate the *average* number of times that a certain amplitude value, x, is reached per second, when the number of zero crossings, f_o, and the RMS-value (σ_x) of the vibration amplitudes are known:

$$f_x = f_o \times e^{-\frac{x^2}{2\sigma_x^2}} \quad (10)$$

It has also been attempted to apply the so-called Poisson probability function to estimate the time elapsing until the amplitude value, x, is exceeded for the first time. It seems, however, that the only proper method of determining the chance for functional failures of the type a) to occur, is to carry out thorough vibration testing of the construction. *Such testing has, furthermore, to be carried out under realistic conditions and at vibration levels which are expected to occur in practice,* i.e. in most cases the testing has to be carried out in the form of a relatively high level, *wide band random vibration test.*

Direct vibration testing may also often be the best approach to detecting functional failures of the type b). However, if the probability distributions for the instantaneous amplitude values of the displacement of critical parts are known, it is also a relatively simple matter to estimate how great a fraction of time a certain displacement amplitude is exceeded. When the probability distribution is *not* known, but the system is supposed to behave linearly, it is possible to obtain the required probability distributions experimentally from vibration tests carried out at a low vibration level.

Conclusion

As will be evident from this lecture a determination of the effects of vibrations on materials and parts today normally requires a combination of theoretical estimates and practical testing.
This is due partly to the fact that our knowledge on the problem of mechanical fatigue is incomplete, and partly to the great complexity of modern mechanical constructions and their responses to the often intricate kind of vibration excitation that occurs in practice. It is hoped that future research in these areas will improve the situation so that more accurate design rules can be laid down than are possible at present.

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Cross Spectral Density Measurements with Brüel & Kjær Instruments. (Part II).

by
Pavel Urban and *Vladimir Kop,*
Motor Car Research Institute, Prague

ABSTRACT

A second version of a practical cross spectral density analyzer is outlined. In this version only one Frequency Analyzer (B & K Type 2107) is needed for the measurements. The measurement method described requires, however, a somewhat more complicated measurement technique than does the method employing two frequency analyzers (Part I).

Following an outline of the basic theory the practical realization of a cross-spectrometer of this type is described. Finally some experimental results obtained by the use of the instrumentation are presented.

SOMMAIRE

Une seconde version d'un analyseur de densité spectrale mutuelle est esquissée. Dans cette version un seul analyseur de fréquence (B & K type 2107) est nécessaire pour les mesures. La méthode de mesure décrite demande cependant une technique de mesure quelque peu plus compliquée que la méthode qui fait appel à deux analyseurs de fréquence (1ère partie).

En suivant un schéma de la théorie fondamentale, on décrit la réalisation pratique d'un inter-spectromètre de ce type. Finalement quelques résultats expérimentaux obtenus par cette méthode sont présentés.

ZUSAMMENFASSUNG

Eine zweite Version eines praktischen Kreuzspektrumsdichte-Analysators wird aufgezeigt. Bei dieser Version wird nur ein Frequenzanalysator (B & K Typ 2107) für die Messung benötigt. Die beschriebene Meßmethode ist jedoch komplizierter als die Methode mit zwei Frequenzanalysatoren (siehe Teil 1).

Nach einer kurzen Darstellung der Grundtheorie wird die praktische Ausführung eines Kreuzspektrometers dieses Typs bescrieben. Schließlich werden einige Versuchsergebnisse vorgelegt, die mit dieser Instrumentierung erzielt wurden.

1. Introduction

In the preceeding B & K Technical Review (No. 3-1968) we have discussed the possibility of cross spectral density measurements by means of two Audio Frequency Spectrometers Type 2112 and some auxiliary equipment. Concluding our discussion we stated that it is not possible to use two Frequency Analysers Type 2107 in the same kind of instrument configuration because identical phase characteristics cannot be guaranteed with this type of tuned analyser. We also mentioned that the 1/3 octave bandwidth might be too broad when cross spectral densities which vary substantially with frequency are measured.

We will now discuss another possibility for cross spectral density measurements using a single analyser only. In this case, however, the method of measurement and evaluation is somewhat more complicated.

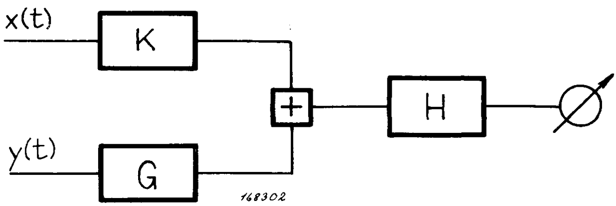

Fig. 1. Basic diagram of a cross spectrometer.

2. Simplified Cross Spectrometer Principle

Fig. 1 shows the block diagram of the second version of a cross spectrometer. The signal, $x(t)$ is passed through the network, K and the signal, $y(t)$ through the network, G. Both output signals are then analogously added and their sum passed through the network, H, formed by a frequency analyser. Its output (signal $x_2(t) + y_2(t)$) is then squared and the RMS value measured.
Mathematically this can be formulated:

$$M^2 = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{+T} [x_2(t) + y_2(t)]^2 \, dt$$

$$= \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{+T} x_2^2(t)\, dt + \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{+T} y_2^2(t)\, dt + \lim_{T\to\infty} \frac{2}{2T} \int_{-T}^{+T} x_2(t)\, y_2(t)\, dt \qquad /1/$$

$$= M_x^2 + M_y^2 + 2\, M_{xy}^2$$

Here M_x can be obtained from a measurement of $x(t)$ only ($y(t)$ not connected), and M_y can be obtained from a measurement of $y(t)$ ($x(t)$ not connected). Thus:

$$M_{xy}^2 = \frac{1}{2}(M^2 - M_x^2 - M_y^2) = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{+T} x_2(t)\, y_2(t)\, dt \qquad /2/$$

The signal, $x_2(t)$ is defined as

$$x_2(t) = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} h(\eta)\, k(\lambda)\, x\,(t - \eta - \lambda)\, d\lambda\, d\eta \qquad /3/$$

and similarly the signal, $y_2(t)$ is

$$y_2(t) = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} h(\delta)\, g(\xi)\, y\,(t - \delta - \xi)\, d\delta\, d\xi \qquad /4/$$

whereby $k(\lambda)$, $g(\xi)$ and $h(\delta)$ are unit impulse responses of the networks K, G and H, and η, λ, δ, ξ are independent time variables.
After substituting the expressions /3/ and /4/ in equation /2/ we can write the following fivefold integral

$$M_{xy}^2 = \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{+T} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} h(\eta)\, h(\delta)\, k(\lambda)\, g(\xi)\, x(t-\eta-\lambda)\, y(t-\delta-\xi)\, d\eta\, d\delta\, d\lambda\, d\xi\, dt \qquad /5/$$

For physically realisable processes and transmission networks we can change the integration sequence and substitute

$$\eta - \delta = \tau$$

$$\lambda - \xi = \vartheta$$

$$\psi_{xy}(\tau - \vartheta) = \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{+T} x(t)\, y(t + \tau + \vartheta)\, dt$$

where $\psi_{xy}(\tau)$ = cross correlation function of the input signals $x(t)$ and $y(t)$. The limits of integration should also be changed. The result, however, makes no difference to real processes. The limits therefore remain unchanged for simplicity.

Equation /5/ then becomes:

$$M_{xy}^2 = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \psi_{xy}(\tau + \vartheta) \left[\int_{-\infty}^{+\infty} h(\tau + \delta)\, h(\delta)\, d\delta\right] k(\vartheta + \xi)\, g(\xi)\; d\xi\, d\tau\, d\vartheta \qquad /6/$$

By means of the double-sided Fourier's transformation the validity of the following relation can be proved

$$\int_{-\infty}^{+\infty} h(\tau + \delta)\, h(\delta)\, d\delta = \int_{-\infty}^{+\infty} H(j\omega)\, H^*(j\omega) \exp(-j\omega\tau)\, df \qquad /7/$$

where $H(j\omega)$ = system function of the network, H

$H^*(j\omega)$ = conjugated function to the system function $H(j\omega)$.

Utilizing this relation and rearranging equation /6/ results in the following expression:

$$M_{xy}^2 = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} H(j\omega)\, H^*(j\omega)\, \psi_{xy}(\tau + \vartheta) \exp[-j\omega(\tau + \vartheta)]. \qquad /8/$$

$$k(\xi + \tau) \exp[j\omega(\xi + \vartheta)]\, g(\xi) \exp[-j\omega\xi]\, d\tau\, d\vartheta\, d\xi\, df$$

which again can be further reduced to yield:

$$M_{xy}^2 = \int_{-\infty}^{+\infty} W_{xy}(j\omega)\, H(j\omega)\, H^*(j\omega)\, K^*(j\omega)\, G(j\omega)\, df \qquad /9/$$

where $W_{xy}(j\omega) = \int_{-\infty}^{+\infty} \psi_{xy}(\tau) \exp[-j\omega\tau]\, d\tau$

Here $W_{xy}(j\omega)$ is the cross spectral density in terms of the correlation function $\psi_{xy}(\tau)$. Equation /9/ above corresponds to equation /12/ in Part 1 of this work.

Provided that the network H is an ideal band pass filter with unity transmission it can be proved that:

$$H(j\omega)\,H^*(j\omega) = \begin{cases} = 1 \;\ldots\; f_d \leqq f \leqq f_h \\ = 0 \;\ldots\; f < f_d; \quad f > f_h \end{cases}$$

where f_d and f_h are the limiting frequencies of the band-pass filter.
Equation /9/ can then be simplified to

$$M_{xy}^2(f) = \int_{f_d}^{f_h} W_{xy}(j\omega)\,K^*(j\omega)\,G(j\omega)\,df \qquad /10/$$

Using the same notation as in part 1 it is possible to define a quantity $M_1(f)$ which is proportional to the real part of the cross spectral density $W_{xy}(j\omega)$ provided that the following relation holds true within the frequency band $B = f_h - f_d$:

$$K^*(j\omega)\,G(j\omega) = 1 \qquad /11/$$

Similarly a quantity $M_2(f)$ may be defined, which is proportional to the imaginary part of $W_{xy}(j\omega)$, if

$$K^*(j\omega)\,G(j\omega) = -j \qquad /12/$$

within the band $B = f_h - f_d$.
If the change of $W_{xy}(j\omega)$ in the frequency band, B, is small we can write the final equation for $W_{xy}(j\omega)$ in the form

$$W_{xy}(j\omega) = \frac{1}{B}\left[\,M_1^2(f) + jM_2^2(f)\,\right] \qquad /13/$$

Utilizing equations /1/, /10/, /11/, /12/ and /13/ we obtain the real part of $M(f)$ as:

$$M_R^2(f) = M_x^2(f) + M_y^2(f) + 2\,M_1^2(f) \qquad /14/$$

and the imaginary part of $M(f)$:

$$M_I^2(f) = M_x^2(f) + M_y^2(f) + 2\,M_2^2(f)$$

The final formula for the cross spectral density then becomes:

$$W_{xy}(j\omega) = \frac{1}{2\,B}\left[M_R^2(f) - M_x^2(f) - M_y^2(f)\right] + \frac{j}{2\,B}\left[M_I^2(f) - M_x^2(f) - M_y^2(f)\right] \qquad /15/$$

3. Cross Spectrometer Realisation – Its Advantages and Disadvantages

In part 1 we discussed cross spectral density measurements using two Audio Frequency Spectrometers Type 2112 to form a cross spectrometer. This type of cross spectrometer has several disadvantages:

I) The measurement necessitates the use of *two* Audio Frequency Spectrometers.
II) The phase characteristics of both band-pass filter sets must be identical.
III) The necessary analog multiplier does not at present belong to general laboratory equipment.

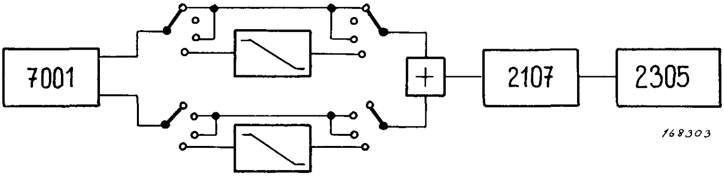

Fig. 2. Block diagram of the spectrometer realization using B & K instruments.

IV) The 1/3 octave bandwidth may in many cases be too wide.
Cross spectral density measurements made according to the principle outlined in section 2 of this article do not suffer from the above disadvantages. The disadvantages of this type of cross spectrometer may be summarized into a single item:

1) To obtain the real (and also to obtain the imaginary) part of the cross spectral density it is necessary to take the difference of three measured quantities. Apart from the "extra" time it takes to measure three instead of two randomly varying quantities, it is obvious that the requirement to measurement accuracy must also be very strict when the differences are to be estimated with a reasonally small error. The complete construction of the cross spectrometer will be affected by this requirement.

The choice of cross spectrometer type to use in a particular measurement situation should actually be decided upon in view of the above discussion.

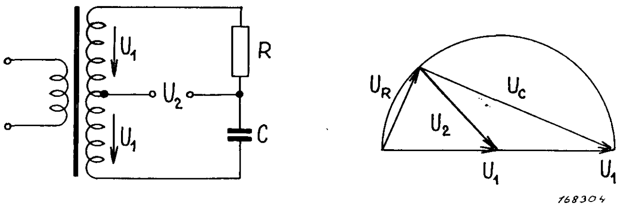

Fig. 3. Principle of operation of the Görges bridge used for phase shifting.

A good solution with regard to the realization of the second type of cross spectrometer is, in the authors' opinion, shown in Fig. 2. Similar to the first type of cross spectrometer (part 1) also this spectrometer makes use of two Görges bridges to provide the required phase shift (90°) in the input signal leads. The principle of operation of the bridges is shown in Figs. 3, 4 and 5. f_m is determined by the equation

$$f_m = \frac{1}{RC} \qquad /16/$$

By a suitable choice of R and C in the two networks used (see also Fig. 2)

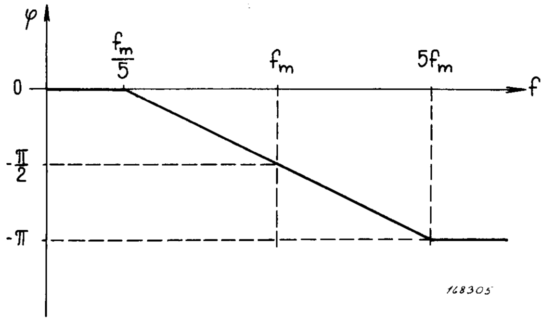

Fig. 4. Theoretical phase characteristic of the Görges bridge vs. frequency.

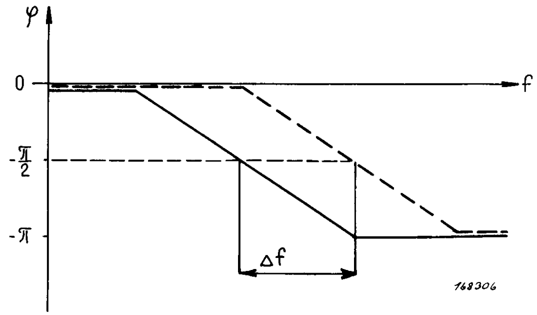

Fig. 5. Theoretical characteristics for two Görges bridges combined to obtain a desired fixed phase difference between two signals.

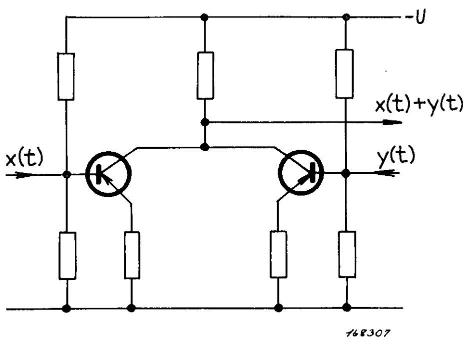

Fig. 6. Basic diagram of the adding amplifier.

the phase angle between the two outputs can be made equal to $\pi/2$ within the frequency band $B = f_h - f_d$, Fig. 5. To add the two outputs an adding amplifier of the type shown in Fig. 6 was used.
In the practical construction of the cross spectrometer both the phase shifting networks and the adding amplifier were built together in one unit. The unit also contained the switches necessary to select the signal paths required for measuring the various quantities defined in section 2 above.

4. Application of the Described Instrumentation.

Spectral Density Measurements.

In part 1 of this work it was mentioned that the most frequent application of cross spectral measurements is the measurement of transfer functions in linear systems. The system may actually involve the transfer of electrical signals in communications, problems regarding servo-circuits and automation, transfer of "biological signals" in the neural system of the human body, transfer of vibrations and stress in mechanical structures, transfer of acoustical energy in complicated acoustical circuits, etc.

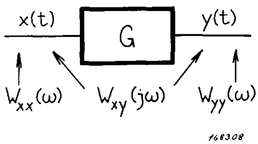

Fig. 7. Notations for the spectral properties of a linear transmission network.

Let us demonstrate one of the very interesting applications of cross spectral measurements in acoustics: The effect of partial noise sources upon the total sound energy at a particular location. Assuming a linear system as shown in Fig. 7, various relations between the inputs and outputs may here be formulated, out of which we select three:

$$y(t) = \int_{-\infty}^{+\infty} g(\tau)\, x(t-\tau)\, d\tau \qquad /17/$$

$$W_{xy}(j\omega) = G(j\omega)\ W_{xx}(\omega) \qquad /18/$$

$$W_{yy}(\omega) = |\,G(j\omega)\,|^2\, W_{xx}(\omega) \qquad /19/$$

Where $x(t)$ = input signal
$y(t)$ = output signal
$g(\tau)$ = unit impulse response of the system
$G(j\omega)$ = system frequency function
$W_{xx}(\omega)$ = spectral density of the input signal
$W_{yy}(\omega)$ = spectral density of the output signal
$W_{xy}(j\omega)$ = cross spectral density of both input and output signal

Assuming that the functions $W_{xx}(\omega)$ and $W_{yy}(\omega)$ represent the frequency distribution of energy at the input and output, respectively, we can write the total energies, P_x and P_y:

$$P_x = \int_{-\infty}^{+\infty} W_{xx}(\omega)\, df$$

$$P_y = \int_{-\infty}^{+\infty} W_{yy}(\omega)\, df$$

The mutual statistical dependence of both signals can be expressed as a function of frequency by means of the cross spectral density. Provided that both signals are uncorrelated one obtains:

$$W_{xy}(j\omega) = 0 \text{ for arbitrary } \omega. \qquad /21/$$

Now consider the system shown in Fig. 8 containing n uncorrelated inputs. The outputs from the separate blocks are added and form a single signal, $z(t)$.

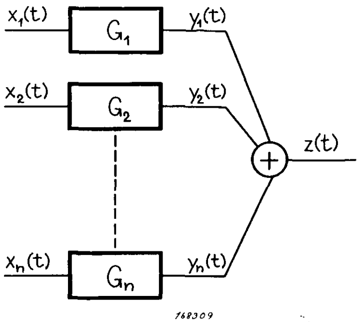

Fig. 8. Schematic illustration of the noise transmission from various sources to an observer's position.

This system is rather general and may, for instance, represent the noise transmission in a mechanical workshop containing a number of different production machines. In such cases we may assume that the separate noise sources are uncorrelated, i.e.

$$W_{x1\,x2}(j\omega) = W_{x1\,x3}(j\omega) = \ldots . = W_{x_{n-1}\,x_n}(j\omega) = 0 \qquad /22/$$

and prove according to (L1) that

$$\left.\begin{aligned} W_{x1\,y1}(j\omega) &= W_{x1\,z}(j\omega) \\ W_{x2\,y2}(j\omega) &= W_{x2\,z}(j\omega) \\ &\vdots \\ W_{x_n\,y_n}(j\omega) &= W_{x_n\,z}(j\omega) \end{aligned}\right\} \qquad /23/$$

The noise power perceived by the observer, P_z, is the sum of the power in the separate signals $y_i(t)$ indicated as P_i

$$P_z = \sum_{i=1}^{n} P_i \qquad /24/$$

This expression can be written out:

$$\int_{-\infty}^{\infty} W_{22}(\omega)\, df = \sum_{i=1}^{n} \left[\int_{-\infty}^{\infty} W_{y_i y_i}(\omega)\, df\right] =$$

$$= \int_{-\infty}^{\infty} \sum_{i=1}^{n} \left[W_{y_i y_i}(\omega)\right] df \qquad /25/$$

i.e.

$$W_{zz}(\omega) = \sum_{i=1}^{n} W_{y_i y_i}(\omega) \qquad /26/$$

That part of the noise power which is received from each separate source can be designated as

$$\alpha_i(\omega) = \frac{W_{y_i y_i}(\omega)}{W_{zz}(\omega)} \qquad /27/$$

whereby

$$\sum_{i=1}^{n} \alpha_i(\omega) = 1 \qquad /28/$$

Now let us return to equations /18/ and /19/. From equation /18/ we obtain

$$G(j\omega) = \frac{W_{xy}(j\omega)}{W_{xx}(\omega)}$$

As $W_{xx}(\omega)$ is a real function, we can write

$$|G(j\omega)|^2 = \frac{|W_{xy}(j\omega)|^2}{W_{xx}^2(\omega)} \qquad /29/$$

Substitution of this expression in equation /19/ reults in

$$W_{yy}(\omega) = \frac{|W_{xy}(j\omega)|^2}{W_{xx}(\omega)} \qquad /30/$$

Referring to equation /23/ we can write for the separate transfer paths, Fig. 8:

$$W_{y_i y_i}(\omega) = \frac{|W_{y_i z}(j\omega)|^2}{W_{x_i x_i}(\omega)} \qquad /31/$$

whereby

$$\alpha_i(\omega) = \frac{|W_{x_i z}(j\omega)|}{W_{x_i x_i}(\omega)\, W_{zz}(\omega)} \qquad /32/$$

By measuring the spectral density of each source signal, the summed "output" signal, and the cross spectral densities of the source and summed "output" we can thus evaluate the participation of each source in the summed "output" power. An example of the result of practical measurements made according

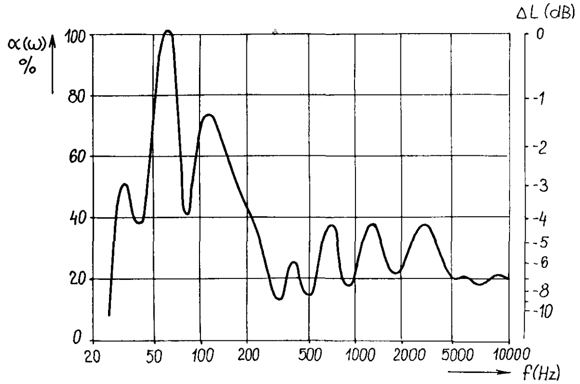

Fig. 9. The results of measurements of the noise power portion $\alpha(\omega)$ of the IC engine in the test room. The measurements were made by means of the described measurement instrumentation.

to the above described technique is shown in Fig. 9. The measurements were in this case carried out on the noise from a single cylinder IC engine placed in a test room where also other noise sources were present. Owing to the fact that the combustion process was here the main noise source the time variation of the cylinder pressure was used as "input" signal. The "output" signal was the sound pressure at a selected measurement position within the test room.

Correlation Function Estimations.

In the theoretical treatment of random processes use is very often made of correlation functions. Now, let us try to determine how cross-correlation functions might be estimated from spectral measurements of the kind discussed in the preceeding text.

As the cross-correlation function is related to the cross spectral density function by means of an inverse Fourier transform (L1, L2, L3) we can write

$$\psi_{xy}(\tau) = \int_{-\infty}^{\infty} W_{xy}(j\omega) \exp(j\omega\tau)\, df \qquad /33/$$

By means of Euler's formula*) for exp. $(j\omega\tau)$ and formula /13/ on p. 24 equation /33/ becomes

$$\psi_{xy}(\tau) = \frac{1}{B} \int_{-\infty}^{\infty} [M_1^2(f) \cos \omega\tau - M_2^2(f) \sin \omega\tau]\, df + \qquad /34/$$

$$+ j \frac{1}{B} \int_{-\infty}^{\infty} [M_1^2(f) \sin \omega\tau + M_2^2(f) \cos \omega\tau]\, df$$

*) Euler's formula: $\exp(j\omega\tau) = \cos(\omega\tau) + j \sin(\omega\tau)$.

For a realizable physical process the function $\psi_{xy}(\tau)$ must be real whereby the imaginary part of equation /34/ must be zero for all values of τ. As, mathematically speaking, we have to consider both positive and negative frequencies the above requirement (a realizable physical process) can be met only when $M_1(f)$ is an even function and $M_2(f)$ is an odd function.
Under these conditions it is possible to write equation /34/ in the form

$$\psi_{xy}(\tau) = \frac{2}{B} \int_0^\infty [M_1^2(f) \cos \omega\tau - M_2^2(f) \sin \omega\tau] \, df \qquad /35/$$

Considering the integral /35/ as a sum of individual integrals, one for each filter band used to measure W_{xy}, /35/ becomes:

$$\psi_{xy}(\tau) = 2 \sum_{i=1}^{n} \frac{1}{B_i} \int_{f_{di}}^{f_{hi}} [M_1^2{}_i(f) \cos \omega\tau - M_2^2{}_i(f) \sin \omega\tau] \, df \qquad /36/$$

$$\psi_{xy}(\tau) = \frac{1}{\pi} \sum_{i=1}^{n} \left[M_1^2{}_i(f) \frac{\sin \omega_{hi}\tau - \sin \omega_{di}\tau}{B_i\tau} + M_2^2{}_i \frac{\cos \omega_{hi}\tau - \cos \omega_{di}\tau}{B_i\tau} \right] \qquad /37/$$

which is the final result of the calculation. The function $\psi_{xy}(\tau)$ thus obtained is approximative and is based on the principle of orthogonal series representation of correlation function measurements mentioned in (L4). To illustrate the use of this technique Fig. 10 shows a plot of the autocorrelation function of automative aggregate vibrations caused by driving the vehicle over a road

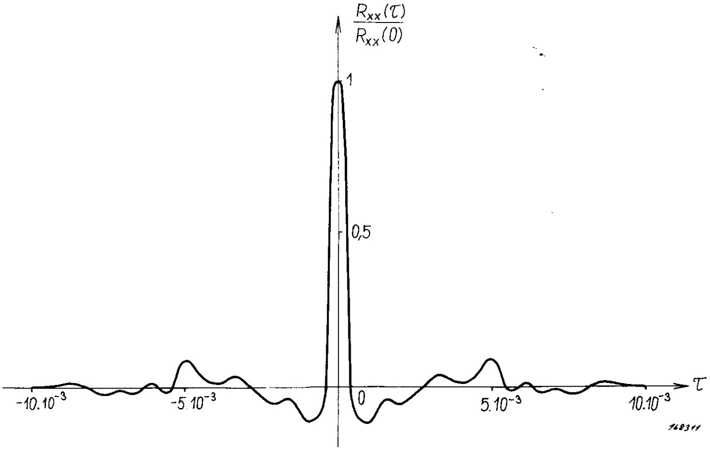

Fig. 10. The estimated auto correlation function of automotive aggregate vibrations.

with rough finish. The correlation function was here calculated from cross-spectral measurements in the manner described above.

5. Conclusion

In the preceeding chapters we have proposed some analog instrumentation for the direct measurement of spectral transmission properties of physical systems. We have also discussed some possible applications for the instrumentation, both with respect to spectral density measurements and with respect to the evaluation of correlation functions.

Concluding the discussion it seems appropriate to briefly summarize some of the areas where cross spectral measurements are, or can be, successfully applied. These are:

Theory of signals	– description of continuous random variables – substitution of random signals with analytically defined functions
Theory of communication	– determination of statistical properties of system outputs by means of known properties of the system – determination of dynamic properties of systems – optimization of dynamic characteristics of transmission elements – proposal of suitable systems
Distribution of electric power	– measurement and evaluation of random changes of mains load
Regulation and automatisation	– determination of transfer characteristics – rejection of effects of spurious signals – optimization of regulation processes
Elasticity and strength	– simulation of random effects in structural load changes – minimalisation of force transmission between dynamically stressed parts
Mechanical engineering	– localisation of sources of quality changes (e.g. of the fibre thickness in textile industry) – surface finish analysis – "diagnostic" method without disassembly
Aeromechanics	– description of turbulence and solution of problems of heat engines (burners, turbines, blowers) – simulation of flow processes around aeroplane surfaces
Hydromechanics	– simulation of effects in the ship building and water engineering – study and description of waves in oceanography
Acoustics	– description of acoustical signals – measurement of transmission losses – measurement of radiation of electroacoustical transducers – analysis of noise sources – measurement of absorption and reflection – description of acoustical fields
Physiology	– analysis of neural signals (EEG)

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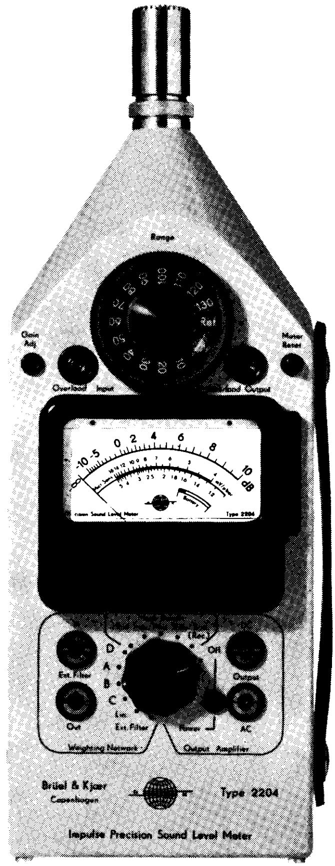
News from the Factory

New Sound Level Meters

The range of noise measuring equipment produced by Brüel & Kjær has now been extended with *three new Sound Level Meters specially designed for use in the field.* Common for all three instruments are their *compact construction,* ease of *application* and *battery operation.* Each of the Sound Level Meters do, on the other hand, comprize their own particular features, some of which are briefly outlined below.

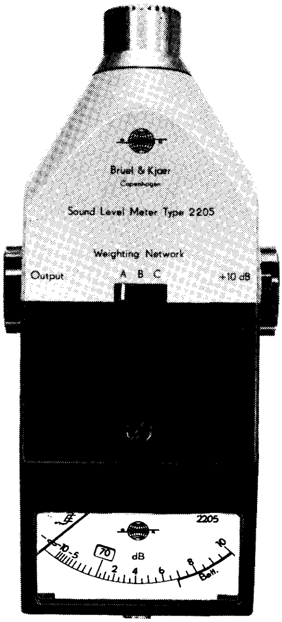
Type 2204 Impulse Precision Sound Level Meter has been designed particularly to meet all the requirements of the new German standard for *impulse sound* level meters (DIN 45633-2) as well as the requirements contained in the corresponding I.E.C.-proposal. Apart from the three standardized weighting network for sound level measurements (A, B and C) the instrument also contains *the new, internationally proposed weighting network (D) for the measurement of aircraft noise.*

The new and improved electronic design used in Type 2204 gives the instrument meter the capability of measuring accurately the RMS value of applied *signals with crest factors as high as 40 (10 at full scale).* A *hold circuit* simplifies the measurement of single impulse sounds and the max. RMS value of any sound of varying level. The very *wide frequency* range of 2 Hz – 70 kHz makes the meter suitable for sound measurements far above the audible range. For remote measurements *a goose neck extension* is supplied and for even more remote mounting the input stage of the instrument can be simply removed with the microphone enabling *a standard B & K microphone extension cable to be used.* This makes the 2204 very suitable for operation under free field conditions.



The Impulse Precision Sound Level Meter Type 2204.

A low inherent noise level allows measurements of *sound levels as low as 15 dB (A)* using the 1″ Microphone Type 4145.
When the Octave Filter Set 1613 is fitted the entire audio range may be frequency analysed. When fitted with the integrator ZR 0020 and one of the B & K accelerometers, the 2204 makes an *excellent portable vibration meter.* 20 *interchangeable scales* (supplied with the instrument) *allow direct readings* of sound level even when different microphone types are used.
The scale system accepts microphone sensitivities in the range 0.4–160 mV per N/m^2 (0.04–16 mV/μbar). Used as a vibration meter the system allows *direct reading of acceleration, velocity and displacement* respectively in m/sec^2, $inch/sec^2$, g, m/sec, m or inch. Accelerometer sensitivities in the range 1–2800 mV/g are accepted. Other features include *overload indicators* for both input and output amplifiers, plug-in unit construction for easy servicing, IEC fast and slow meter damping, DC and AC outputs and built-in facilities for calibration and battery check.



The Sound Level Meter Type 2205.

Type 2205 Miniaure Sound Level Meter is an instrument designed for accurate sound level measurements in accordance with national and international standards. Its *small size* and *simple operation procedure* makes it the ideal tool for the person with a need to make sound measurements without being a specialist in acoustics. In the standard version, equipped with a piezo-electric microphone, *the instrument fulfils the usual standards for ordinary sound level meters.*

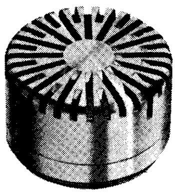
Type 2206 Miniature Precision Sound Level Meter is a special version of the above described Type 2205 Sound Level Meter which has then been supplied with a B & K 1/2″ Condenser Microphone (Type 4148) and a special microphone adaptor. It gives exact *sound level measurements in accordance with the full precision standards* (I.E.C. Recommendation Publication 179).
Like Type 2205 it is extremely easy to operate:
All attenuators are operated by a single switch which sets the measuring range. The setting appears in a window on the meter scale giving direct

reading of the sound level. The range control can be locked in any of its ten positions (30 to 120 dB) and a separate push-button switch can be used to quickly increase the measuring range by 10 dB, regardless of the setting of the range control.
An output is provided for feeding external recording instruments such as a B & K Level Recorder 2305 or a magnetic tape recorder. The instrument is powered with an ordinary 1.5 V cell, providing 10 hours of operation under normal use.

The Precision Sound Level Meter Type 2206 with Wind Screen UA 0082.

Piezoelectric Microphone Type 4117

This one inch diameter microphone is designed primarily for use with the B & K sound 'evel meters, but will be found useful as a *high quality, low cost measuring microphone* in various applications such as sound distribution compensating systems in theatres, public address systems, machinery monitoring systems in factories, power plants etc.
Because of the high capacity of the microphone the input impedance of the amplifier need not be very large, e.g. a 2 MΩ impedance gives a low frequency cut-off at about 20 Hz. Relatively long connection cables may be used between microphone and preamplifier without serious loss of sensitivity.



The Piezoelectric Microphone Type 4117.

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Brüel & Kjær

ADR.: BRÜEL & KJÆR
NÆRUM - DENMARK

TELEPHONE: 800500
BRUKJA, Copenhagen

TELEX 5316

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